# Philosophy 151: Definitions from *van Dalen*

Lucas Garron

March 9, 2011

Trying to make sense of a "slightly" inconsistent book about logic.

## Chapter 1. Propositional Logic

*PROP* **(7)** smallest set $X$ with: (i) $p_i \in X$
– $\bot \in X$
(ii) $(\varphi \Box \psi) \in X$ $\{\wedge, \vee, \rightarrow, \leftrightarrow\}$
(iii) $(\neg\varphi)$

Typical element name: $p$

**rank (12)** Recursively defined "depth" of a proposition.

**valuation (18)** A mapping $v : PROP \rightarrow \{0, 1\}$, defined recursively on $PROP$. $[|\varphi \wedge \psi|] = v(\varphi \wedge \psi) = min(v(\varphi), v(\psi))$, etc.

$\models \varphi$ **(19, propositional)** $\varphi$ is a tautology (true for all valuations)

$\Gamma \models \varphi$ $[|\psi|]_v = 1$ or all $\varphi \in \Gamma \Leftrightarrow [|\varphi|]_v = 1$

$\Gamma \vdash \varphi$ **(36)** There is a derivation with conclusion $\varphi$ and all hypotheses in $\Gamma$

$\vdash \varphi$ **(36)** $\Gamma = \emptyset$. $\varphi$ is a theorem.

**Consistency** $\Gamma$ (set of propositions) is consistent if $\Gamma \nvdash \bot$.

**Maximal Consistency** $\Gamma$ consistent such that $\Gamma \subseteq \Gamma'$ and $\Gamma'$ consistent $\Rightarrow \Gamma = \Gamma'$.

## Chapter 2. Predicate Logic

**Structure (58)** An ordered sequence $\langle A, R_1, ..., R_n, F_1, ..., F_m, \{c_i | i \in I\}\rangle$, where the relations and functions are on $A$, and the constants are elements of $A$.

Typical element name: $\mathfrak{A}$

**Similarity Type (of a Structure) (59)** A sequence $\langle r_1, ..., r_n; a_1, ..., a_n; \kappa\rangle$ where $r_i$ an $a_i$ are arities (number of arguments / number of arguments without output value) of $R_i$ and $F_i$, and $\kappa$ is the number of constants.

Typical name: none.

**Language** A set of expressions(?) (sentences?) built up of a set of symbols with amounts corresponding to a similarity type.
(n.b. $\doteq$ is always a relation: identity/equality.)

Typical name: $L$

**Universe of a Structure** $|\mathfrak{A}| = A$ as in the definition.

$TERM$ **(61)** smallest set $X$ such that:
(i) constants $\overline{c}_i, i \in I$, variables $x_i, i \in \mathbb{N}$
(ii) $t_1, ..., t_{a_i} \in X \Rightarrow f_i(t_1, p..., t_{a_i}) \in X$

Typical element name: $t$.

$FORM$ **(61)** smallest set $X$ with:
(i) $\perp$
- - $P_i(...)$ (... each in X)
- - $t_1 = t_2 \in X$
(ii) $(\varphi\Box\psi) \in X$ $\{\wedge, \vee, \rightarrow, \leftrightarrow\}$
(iii) $(\neg\varphi)$
(iii) $((\forall x_i)\varphi)$, $((\exists x_i)\varphi)$

Typical element name: $\varphi$

**Free variables (63-64)** The set $FV(t)$ and $FV(\varphi)$ is defined recursively.

**Closed / Open / Sentence (64)** $t$ (or $\varphi$) is closed if $FV(t) = \emptyset$.
A closed formula is a sentence.
A formula without quantifiers is open.

$TERM_c$ **(64)** Set of closed terms (in a language?).

$SENT$ **(64)** Set of sentences (in a language?).

**Free variable (66)** $\varphi$ free for \$ in $\varphi$, defined recursively using $FV$.

**Extended language (67)** $L(\mathfrak{A})$: add constant symbols for all elements of $\mathfrak{A}$ to $L$. $\overline{a}$ from $a \in |\mathfrak{A}|$

**Interpretation/valuation of sentences in $L(\mathfrak{A})$ (69-70)** Recursively defined valuation over $FORM$(?).
$[|\varphi|]_{\mathfrak{A}}$ or $v_{\mathfrak{A}}(\varphi)$ is a mapping $[|.|]_{\mathfrak{A}} : SENT \rightarrow \{0, 1\}$, recursively defined.
(Valuation of a term: $(.)^{\mathfrak{A}} : TERM_c \rightarrow |\mathfrak{A}|$)

$\mathfrak{A} \vDash \varphi$ **(70)** Defined as $[|\varphi|]_{\mathfrak{A}} = 1$ ($\vDash$ is the "satisfaction relation").

**(Universal) closure** If $FV(\varphi) = z_1, ..., z_k$, then $Cl(\varphi) := \forall z_1 ... z_k \varphi$.

**Some Semantics (71)** (i) $\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{A} \models Cl(\varphi)$
(ii) $\models \varphi \Leftrightarrow \mathfrak{A} \models \varphi \ \forall \mathfrak{A}$ (of the appropriate type)
(iii) $\mathfrak{A} \models \Gamma \Leftrightarrow \mathfrak{A} \models \psi \ \forall \psi \in \Gamma$.
(iv) $\Gamma \models \varphi \Leftrightarrow (\mathfrak{A} \models \Gamma \Rightarrow \mathfrak{A} \models \varphi)$, where $a \in \Gamma$ and $\varphi$ are sentences.

**Model (71)** If $\mathfrak{A} \models \sigma$, then $\mathfrak{A}$ is a model of $\sigma$.

**Semantic consequence (71)** $\varphi$ is a semantic consequence of $\Gamma$ if $\Gamma \models \varphi$ ($\varphi$ holds in each model of $\Gamma$.)
($\models \varphi$ means $\varphi$ is *true*.)

**Satisfiability** A formula with free variables $z_i$ is satisfiable if there is a set of elements $a_i \in |\mathfrak{A}|$ such that $\mathfrak{A} \models \varphi[\overline{a}_1, ..., \overline{a}_k, /\overline{z}_1, ..., \overline{z}_k]$ (substitution).

**Prenex (normal) form (78)** $\varphi$ with quantifiers followed by an open formula.
(Theorem: For each $\varphi$, $\exists$ a prenex formula $\psi$ such that $\models \varphi \leftrightarrow \psi$)

**Identity Axioms** $I_1$: $\forall x \ (x = x)$ (reflexive)
$I_2$: $\forall xy \ (x = y \rightarrow y = x)$ (symmetric)
$I_3$: $\forall xyz \ (x = y \wedge y = z \rightarrow x = z)$ (transitive)
$I_4$: If the arguments are equal, $t(x_1, ..., x_n) = t(y_1, ..., y_n)$ and $\varphi(x_1, ..., x_n) \rightarrow \varphi(y_1, ..., y_n)$

## Chapter 3. Completeness

**Theory (104)** A theory $T$ is a collection of sentences such that $T \vdash \varphi \Rightarrow \varphi \in T$ (closed under derivability).
Typical name: $T$.

**Axiom Set (104)** A set $\Gamma$ such that $T = \{\varphi | \Gamma \vdash \varphi\}$.

**Henkin Theory (104)** $T$ is a Henkin theory if for each sentence $\exists x \ \varphi(x)$ there exists a constant $c$ such that $((\exists x \ \varphi(x)) \rightarrow \varphi(c)) \in T$. ($c$ is a *witness* for $\exists x \ \varphi(x)$.)

**Extension (of a theory) (104)** Theories $T$ and $T'$ with respective languages $L, L'$:
(i) $T$ is an extension of $T'$ if $T \subseteq T'$
(ii) $T'$ is a *conservative extension* of $T$ if $T' \cup L = T$

$L*$: Add $c_\varphi$ for each $\varphi$ of the form $\exists x \ \varphi(x)$
$T*$: $T \cup \{\exists x \ \varphi(x) | \exists x \ \varphi(x)$ closed, with witness $c_\varphi\}$ (theorem: conservative over $T$)

**Model Existence Lemma (103, 109)** A theorem. If $L$ has cardinality $\kappa$ and $\Gamma$ is a set of consistent sentences, then $\Gamma$ has a model of cardinality $\leq \kappa$.

**Compactness Theorem (111)** $\Gamma$ has a model $\Leftrightarrow$ each finite $\delta \subset \Gamma$ has a model.

$Mod(\Gamma)$ $Mod(\Gamma) = \{\mathfrak{A} | \mathfrak{A} \models \sigma$ for all $\sigma \in \Gamma\}$.

**Theory of** $(\mathcal{K})$ If $\mathcal{K}$ is a class of structures with the same similarity type, $Th(\mathcal{K}) = \{\sigma | \mathfrak{A} \models \sigma \text{ for all } \mathfrak{A} \in \mathcal{K}\}$

**Reduct / Expansion** $\mathfrak{A}$ is a reduct of $\mathfrak{B}$ if $|\mathfrak{A}| = |\mathfrak{B}|$ and $R_i, F_j, c_k$ from $\mathfrak{A}$ are also in $\mathfrak{B}$. $\mathfrak{B}$ is an expansion of $\mathfrak{A}$.

**Axiomatizability** A class $\mathcal{K}$ of structures is (finitely) axiomatizable of there is a (finite) set $\Gamma$ such that $\mathcal{K} = Mod(\Gamma)$.

**Structure Universe Homomorphism (119)** (i) $f : |\mathfrak{A}| \to |\mathfrak{B}|$ is a homomorphism if each $P_i, F_j, c_k$ maps from $\mathfrak{A}$ to $\mathfrak{B}$ if $f$ is mapped over its arguments.
(ii) $f$ is an isomorphism if it's also bijective and predicates can map back.

**Isomorphic Structures (119)** $\mathfrak{A} \cong \mathfrak{B}$ if there is an isomorphism $f : \mathfrak{A} \to \mathfrak{B}$.

**Elementary Equivalence (NOT SYMMETRIC) ($\mathfrak{A} \equiv \mathfrak{B}$) (119)** $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$ if for all sentences $\sigma \in L$ (language of $\mathfrak{A}$), $\mathfrak{A} \models \sigma \Leftrightarrow \mathfrak{B} \models \sigma$. (Note: $\mathfrak{A} \equiv \mathfrak{B} \Leftrightarrow Th(\mathfrak{A}) = Th(\mathfrak{B})$)

**Substructure / Submodel ($\mathfrak{A} \subseteq \mathfrak{B}$) (119)** $\mathfrak{A}$ is a substructure/submodel of $\mathfrak{B}$ (same type) if all elements of $\mathfrak{B}$, "restricted to the universe of $\mathfrak{A}$," are in $\mathfrak{A}$.

**Elementary Substructure ($\mathfrak{A} \prec \mathfrak{B}$) (119)** $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$ ($\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$) if $\mathfrak{A} \subseteq \mathfrak{B}$ and for all $\varphi(...) \in L$, $a_i \in |\mathfrak{A}|$, $\mathfrak{A} \models \varphi(\overline{a}_1, ..., \overline{a}_n) \Leftrightarrow \mathfrak{B} \models \varphi(\overline{a}_1, ..., \overline{a}_n)$. ($\mathfrak{A}$ and $\mathfrak{B}$ have the same true sentences *with parameters in* $\mathfrak{A}$.) Note that $\mathfrak{A} \prec \mathfrak{B} \Rightarrow \mathfrak{A} \equiv \mathfrak{B}$.

**Complete theory (124)** $T$ with axioms $\Gamma \subset L$ is called complete if for each sentence $\sigma \in L$, either $\Gamma \vdash \sigma$ or $\Gamma \vdash \neg\sigma$.

**$\kappa$-categorical (125)** Let $\kappa$ be a cardinal. $T$ is $\kappa$-categorical if it has exactly one model of cardinality $\kappa$ up to isomorphism.

**Model complete (131)** $T$ is model complete if $\mathfrak{A}, \mathfrak{B} \in Mod(T)$, $\mathfrak{A} \subseteq \mathfrak{B} \Rightarrow \mathfrak{A} \prec \mathfrak{B}$.

**Prime model** $T$ has a a prime model if that model is contained in every model of $T$ up to isomorphism.

## Chapter 4. Second Order Logic

**Second-order alphabet** (i) individual variables $x_0, ...$
(ii) individual constants $c_0, ...$
for each $n \geq 0$:
(iii) $n$-ary set (predicate) variables $X_0^n, X_1^n, ...$
also for each $n \geq 0$:
(iv) $n$-ary set (predicate) constants $\bot, P_0^n, P_1^n, ...$
(v) connectives: $\wedge', \to, \vee, \neg, \leftrightarrow, \exists, \forall$.
(and auxiliary symbols: (),)
Countable variables of each kind, any number of constants.

**Second-order formulas** $FORM$ is inductively defined, again:
(i) $X_i^0, P_i^0, \bot \in FORM$
(ii) for $n > 0$, $X^n(t_1, ..., t_n) \in FORM$, $P^n(t_1, ..., t_n) \in FORM$
(iii) $FORM$ is closed under the propositional connectives
(iv) $FORM$ is closed under first- and second-order quantification.

**Second-order structure (144)** $\mathfrak{A} = \langle A, A*, c*, R* \rangle$, where:
$A* = \langle A_n | n \in \mathbb{N} \rangle$
$c* = \{c_i | i \in \mathbb{N}\} \subset A$
$A = \langle R_i^n | i, n \in \mathbb{N} \rangle$, and $A_i \subseteq \mathcal{P}(A^n), R_i^n \in A_n$.

**Full structure (144)** $A_n = \mathcal{P}(A^n)$; each $A_n$ contains all $n$-ary relations.

**Validity (144)** $\mathfrak{A} \models \varphi$ similar to first-order logic.

**Comprehension Schema (145)** $\exists X^n \forall x_1 ... x_n [\varphi(x_1, ..., x_n) \leftrightarrow X^n(x_1, ..., x_n)]$.

**Model of second-order logic (147)** A second-order structure $\mathfrak{A}$ is a model of second-order logic if the comprehension schema is valid in $\mathfrak{A}$.

## Intuitionistic Logic

**Gödel Translation** The mapping $^\circ : FORM \to FORM$:
(i) $\bot^\circ := \bot$, $\varphi^\circ := \neg\neg^\varphi$.
(ii) $(\varphi \wedge \psi)^\circ := \varphi^\circ \wedge \psi^\circ$
(iii) $(\varphi \vee \psi)^\circ = \neg(\neg\varphi^\circ \wedge \neg\psi^\circ)$
(iv) $(\varphi \to \psi)^\circ := \varphi^\circ \to \psi^\circ$
(v) $(\forall x\ \varphi(x))^\circ := \forall x\ \varphi^\circ(x)$
(vi) $(\exists x\ \varphi(x))^\circ := \neg\forall x\ \neg\varphi^\circ(x)$

(Theorem: $\Gamma \vdash_c \varphi \Leftrightarrow \Gamma \vdash_i \varphi^\circ$)

**Kripke Model** $\mathcal{K} = \langle K, \Sigma, C, D \rangle$. $K$ is a (non-empty) poset, $C$ a function on the constants of $L$, $D$ a set-valued function on $K$, $\Sigma$ a function on $K$ with certain constraints. ($D$ and $\Sigma$ also satisfy constraints.)

## Comments

Not included:

1. Some alphabets
2. Various definitions of substitution.
3. $Diag(\mathfrak{A})$, isomorpically embedded (120).
4. decidable ($\Gamma$), decidable ($T$), effectively enumerable ($\Gamma$), effectively axiomatizable ($T$)
5. Skolem functions, axioms, extensions, expansions (136), hulls (141)