MATH163: FINAL PAPER

On the Role of Unformalized Geometric Reasoning in Apollonius's Conics

[Or: All those little things that bother mathematicians about Greek geometry proofs.]

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NOTE: This paper is a commentary based on Taliaferro's English translation of (Book I of) Apollonius's *Conics* and the proofs within. I will reason based on the assumption that it is a faithful translation of the extant Greek, and that the reader is either familiar with the work, or has the book available for reference about details.

Introduction

Greek mathematicians are known by the results they proved about important geometric shapes and processes. While Euclid is known for ruler-and-compass constructions, Pythagoras is the namesake of a certain theorem about right triangles, and Archimedes proved a large variety of geometric and physical results, the lesser-known Apollonius of Perga is best remembered for his book on the *Conics*. In the *Conics*, he begins by showing the proportional relationships arising from all the possible sections produced by slicing a cone, and uses this to motivate his new names for the shapes: *parabola*, *hyperbola*, and *ellipse*. He goes on to prove a thorough set of results about the properties of these curves using the traditional level of formalism in Greek mathematics — which, however, occasionally includes questionable leaps of logic such as the use of diagrams in supporting assertions. While this does not undermine the importance or integrity of the work, it is instructive to look at such proofs from a modern perspective.

In many ways, the best Greek mathematicians had "the right idea". Their results are still valid today, and their overall developments hold up to scrutiny. Modern mathematical reasoning has had a long development, but many of the important foundational ideas can be traced down to the Greeks: Euclid is famous for his early treatment of geometry by laying down assumptions and successively proving theorems from them — and others like Apollonius adopted the same approach. At the beginning of the 20th century, this idea was developed into the formalization of mathematics, which justifies the correctness of a mathematical system based on axioms, proofs, and theorems. But while the Greeks had the right idea, many of their proofs use leaps of logic that would not allowed in a normal formalism; it was not until the late 19th century that David Hilbert developed a proper formalization of Euclidean geometry, expanding Euclid's 5 postulates into 21 axioms that fill in the implicit assumptions in Euclid's reasoning. Indeed, modern explorations of Greek proofs focus on Euclid, and the extent to which his proofs rely on relations in specific diagrams to draw general conclusions. However, we can try to apply some of these concerns to Apollonius, who had the difficulty of working with more general (and less intuitive) curves in the *Conics*.

In particular, propositions 22-36 in book I of the *Conics*, Apollonius develops several conditions for lines to be tangent or intersect conic sections. The proofs depend on being able to draw specified (actually possible or hypothetical) points on diagrams, and using them to arrive at relations. While it is clear that the proofs do "the right thing" in that they calculate proportions accurately, there are parts where he glosses over topological considerations or condenses multiple cases implicitly.

A Demonstration of Common Concerns

To see the issues with Apollonius's reasoning, we will closely inspect Proposition I.24, which is simple, but highlights topological, diagrammatic, and logical concerns that occur are relevant throughout the *Conics*:

> *If a straight line [CDE], meeting a parabola or hyperbola at a point [D], when produced both ways, falls outside the section, then it will meet the diameter [AB].*

The result is proved by two main steps:

- Add a point F on the section, and note that DF must meet AB (by a literal application of I.22).
- Since CDE lies between the section and FDA, it must also meet the diameter AB outside the section.

Let's address a few geometrical ideas used in the proof:

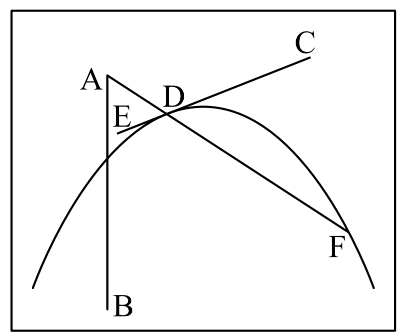

Diagram for the proof of I.24. Note that AB is the diameter, which is actually centered on the section, but not drawn so.

Concern 1: "Outside the Section"

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First, it is interesting to note that the concept of "outside the section" is surprisingly well-defined. The Jordan curve theorem shows that the notion of regions "inside" and "outside" a curve are tricky to capture, and it seems like this could only be worse for an infinite conic section. However, the outside of a conic section is defined by the exterior of the original cone, which has a relatively clear definition. (In fact, one way to view Apollonius's goal with propositions like these is to provide useful rules for working with the boundary between the interior and exterior of the section.)

Concern 2: Respecifying Points

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Apollonius sometimes modifies underspecified points to have some specific role. In this case, A and B were two arbitrary points defining the diameter line. They are underdetermined, so, so he "respecifies" A by assuming the place that FD meets AB is A. While questionable, it seems plausible that this reuse of A is simply a notational convenience to make diagrams and proofs simpler by using fewer letters. He *could* use new letters, as a modern correct proof would require, but the proofs do not lose any correctness if we treat this as a shorthand. Of course, this only holds if his proofs are careful not to rely on impossible specifications (e.g. assigning a point to hold two roles that are contradictory) or invalid assumptions (that a certain locus can exist before this is proven, or that two points intended to define a line actually coincide). However, most respecifications of points involve assigning specific locations of points like A in I.24, and sometimes he leaves points in place (e.g. point F in I.26). Thus, the proofs make it clear that Apollonius is aware of this, and avoids misusing the convenience or respecifying.

Concern 3: Representing the General Case

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Although respecifation in Concern 2 may be a valid application of the phrase "without loss of generality", one of the main concerns in the literature critical about Greek proofs is that they appear to rely on proving relationships about specific diagrams accompanying them, often relying critically on the diagrams to refer to parts of the construction in the proof. Although the diagram in I.24 is careful to avoid assuming undue consequences of symmetry by placing AB off-center, the initial diagram and the proof make other tacit choices:

✗ Concern 3.1: CDE lies to the right of AB (i.e. D touches to the section the right of AB).

In this case, it is easy to see that the proof is just as relevant, regardless of the side CDE touches the section. However, this highlights a special case: What if D is *on* the diameter?

✗ Concern 3.2: D is not on AB.

In this case, meeting "on the diameter" would be trivially satisfied. However, the region between AB, AD, and the section used in the proof *does not exist*, and other degenerate results related to tangency at the vertex, are not always covered by the *Conics*. This could be modified by adding a clause to the theorem and/or handling it separately, but those two options are less general and inelegant.

✗ Concern 3.3: F lies to the right of D.

This is a particularly interesting assumption. It is clear that the proof makes no sense unless F is to the right of D, but Apollonius never specifies that F is to the right, relying on the diagram to convey it. Perhaps this is because describing an (arbitrary) construction "to the right of D" is more trouble than it's worth, but this is a significant concern.

✓ Concern 3.4: [C]DE lies between the section and [F]DA, i.e. A lies above CDE

Although this may seem nitpicky, the proof depends on A lying above CDE. However, without specific locations, or advanced knowledge about section in question (e.g. I.24 itself), it is conceivable that A lies below CDE. However If we assume that the section lies entirely to one side of CDE, then point F must be on that side. By Proposition I.22, FD meets the diameter, and *if A is on the extension of FD in the direction of D*, then A lies on the other side of CD. Given the assumption that D lies to right of AB, and the fact that the outside of the section and the relevant side of FDE are defined, this deduction is relatively sound.

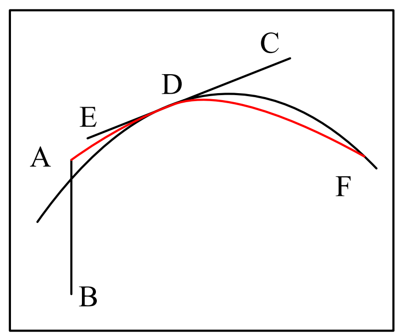

Diagram gone wrong.
Note that FDA is still a "line" to the Greeks (just not a "straight line").

Concern 4: Missing Axioms

Many potential issues with Apollonius's proofs are hard to spot because they skip over reasoning that is completely intuitive, but not as simple as intuition would suggest. In an algebraic proof, it is easy to justify skipping steps because "the math works out", but this is tricky in geometry when the statements are not formalized, and one cannot assume an implicit translation into the proper steps. This proof makes such a leap when it states that CDE will meet the diameter (outside the section).

At first glance, it seems obvious: CDE enters the region bounded by {A, D, the vertex of the section}. It enters at CDE, and cannot cross FDA again, nor touch the section again, so it has to exit through AB.

However, the truth of such a statement ultimately has to be justified by axioms. Of course, Apollonius didn't list any axiom to address this explicitly, but it does not seem likely that he considers there to be an implicit one that is relevant. And while it seems that the assertion here can be taken apart and shown through more fundamental steps, it is the modern view that the Greeks fundamentally missed something in their premises.

In fact, Hilbert introduces a very similar axiom in his formalization of Euclidean geometry:

> *Foundations of Geometry, Axiom II.5 (original numbering): Let A, B, C be three points not lying in the same straight line and let a be a straight line lying in the plane ABC and not passing through any of the points A, B, C. Then, if the straight line a passes through a point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC.*

Although this is more like a converse of the reasoning Apollonius uses here, the basic assertion is that a (straight, extended) line entering a triangle will exit it in an expected way. In 1882, Pasch proved that this axiom is not provable from Euclid's axioms. In other words, this intuitive leap in the proof of I.24 *cannot* formally be resolved by the approach that the Greeks used. (Even worse, the region in I.24 has a curved side, which means that an axiom/theorem about a triangle does not directly apply here! The proof would have to be modified to introduce an actual triangle, or to have extended assumptions.)

In addition, it is notable that Hilbert emphasized clarifying certain relations between objects in his formalization; in contrast, the concept of *between* (and therefore *inside*) is never clearly characterized by the Greeks (arguably, it also isn't by Hilbert, but at least Hilbert's axioms specify exact rules for reasoning with it).

Therefore, Apollonius's reasoning is faulty as presented. While it is clear that line CDE does need to "reach the other side" of AB and cross it outside the section, it is *not* clear if a simple modification of the proof would avoid the need for a new (at least implicit) axiom. We could try to support this step using semantic reasoning, but this is tricky: we would need to show that there is no valid model embodying Apollonius's explicit and implicit axioms in which CDE does not cross AB outside the section. However, this either reduces to proving the original claim (in which case the semantic argument is unnecessary), or it would have to appeal to some meta-principle. This is reminiscent of Euclid's parallel postulate, where the intuition is at odds with a valid non-standard approach.

In practice, since we would like to ensure the theorem is formally correct (the line CDE is not parallel to AB), a good resolution might be to modify the proof so that an axiom like Hilbert's II.5 can be invoked directly, or take a different approach and show that CDE is not parallel to AB (thus ensuring an intersection point). I.26 may be applicable, and is independent from I.24 ,but does not directly help.

The Euclidean Paradigm and Diagrams

Most analyses of Greek mathematical treatises focus solely on concerns about Euclid's Elements. This is for a good reason; Euclid is responsible for setting down the most basic levels of geometry, and having the most prominent focus on introducing definitions and postulates formally. In addition, important later works generally follow the framework of Euclid, presumably using his rules and assumptions supplemented with definitions for new objects of study (like conic sections). Thus, it is important to "get Euclid right" first, before "getting Archimedes right" or "getting Apollonius right".

There are roughly two approaches to "getting Euclid right". The first has already been discussed: Introduce a new system to supersede Euclid's definitions, in which his theorems hold according to some valid model. This approach has only become relevant in the early 20th century, motivated by Hilbert's program. However, Hilbert published his revision of Euclidean geometry *before* he developed his program! Therefore, Hilbert's *Foundations* uses Euclidean geometry is a seminal example of such revision. Further, it is also widely accepted that Hilbert's is the first successful attempt at this (although other mathematicians have offered alternatives).

The other approach is to fix Greek mathematics by supplementing the existing works with sufficient fixes, i.e. actually "making Euclid right". Saccheri's Renaissance attempt to have "Euclid freed of all blemish" demonstrates a (slightly misguided) attempt at this. More recently, people have tried to use mature logical notions of a formal system to characterize Euclidean geometry. Isabel Luengo provided a diagrammatic system for a subset of Hilbert's geometry (*Allwein and Barwise*, Chapter VII). More recently, John Mumma has tried to provide a system to support *all* of the reasoning behind Euclid's original work. Both works focus largely on the issue of diagrams: while diagrams are supposed to *complement* a general argument by giving visual intuition, Greek geometry ultimately relies on the specifics of those diagrams for reasoning. For example, trying to justify the introduction of certain points through pure syntactic rules in a written proof is impractical, and obscures the essential insights of the original proofs. If we want Euclid to be *right*, we have to accept the use of diagrams for reasoning. As Allwein and Barwise points out, this is at best unconventional — and contrary to Hilbert's original view (Mumma, pg. 142).

This brings up an issue: how can we *represent* and reasoning with diagrams in a manner that can be justified formally? Luengo and Mumma try to show that it is possible to describe diagrams in such a way that they capture all the information specified, say, in a theorem, but such that incidental information (e.g. the exact location of D in I.24) is not. Mumma goes in depth to describe a system **Eu** that preserves what he describes this, by defining isomorphisms over *co-exact* features. This allows Euclid's proofs to be formalized quite faithfully, but sometimes requires tidying up implicit parts of proof, e.g. splitting a diagrammatic proof into multiple cases.

Concerns of the *Conics*

Although Luengo's and Mumma's ideas attempts to cover much / all of Euclidean geometry, they require a lot of meticulous work to patch Euclid's vague rules and postulates into acceptably modern axioms. Although the ideas are credible, they do not reduce the issue of justifying original Greek geometric arguments into a simple task, and still leave it tricky to reconcile metric relationships with "non-specific" diagrams.

In the case of Apollonius, we have more complicated curves than in the Elements (although, interestingly, conic sections are derived from cones — which *are* based on circles), which go a little beyond the simplicity of circles that feature in Mumma's arguments. For example, *extensible* lines are straightforward, but unlike circles, (non-elliptic) unbounded conic sections have less straightforward propertires. We have the issues of Euclidean geometry, *and then some*.

Fortunately, much of the *Conics* (especially Book I) are not actually concerned about construction as much as characterizing useful properties of conic sections. Much of this actually relies on the proportions that give the parabola, hyperbola, and ellipse their names; in essence, the diagrams rely on establishing points that produce relationships/proportions that resolve into desirable result (e.g. conditions for tangency). In this sense, much of the truth of the statements lies in the algebra (sometimes, as in I.34, pages of it!) that comes from segments related to conics. Perhaps such a statement is true of much of Greek mathematics, but it is surprisingly relevant to the reasoning where Apollonius relies on diagrams: the diagrams just give us the algebra! Often, the algebra is even the same for different curves and different ways of extending the diagram (e.g. in some proofs, changes like allowing P to lie to the left of D in the diagram for I.24, actually work).

So why not replace all of it with algebra? As some authors have pointed out, the ordinates and the diameter of a conic section provide an ad-hoc coordinate system for metric relationships. We could simple place the sections on a full coordinate plane, and calculate Apollonius's relationships and properties of conics and lines that way. However, at that stage, we begin emulating Hilbert's second chapter in the *Foundations* by providing a model for the geometry. Further, proofs would either rely on some similarly unformalized reasoning with diagrams, and arguments about tangency in particular would probably need to resort to calculus. Any attempt to make the algebra more palatable, formal, and modernized, would be inclined to move even further from Apollonius's proofs, which defeats the point of exploring the notions in the original *Conics*.

In the end, two important concerns come out in the case of the *Conics*: more general tangents than those to circles (the overall provability of I.24), and a strong reliance on introduction lines and points to establish segments with useful proportions (i.e. "Concern 3.3"). Other concerns, like extra/degenerate cases ("Concern 3.1/3.2") and topology ("Concern 4") are just as relevant in Euclidean geometry, and therefore not specific to Apollonius.

In the case of tangents, it seems that the actual proofs are "hopelessly lost" from a formal perspective. In order to save them, we would either have to introduce new assumptions to prove them, take some assumptions about tangency. Once we *have* results about tangency/non-tangency, we can use them elsewhere. Thus, this issue reduces mostly to a semantic justification for proofs like I.24.

The introduction of points and lines is a more general problem: How can a proof be correct if it relies on general relationships that may not always hold? We can often consider the diagrams to have multiple cases where a point might have been introduced in multiple places, and/or we could modify the text to have more accurate specifications about the locations of new points, but it is not necessarily straightforward to make this work for the *Conics*. In particular, it is also unclear how to deal with counterfactual diagrams (e.g. in I.34). We could try to show that the proofs can be decomposed into "safe" separate cases, but it would take some work to show what leads to which cases, and justifying that certain proportions remain holds us accountable to algebra again. It seems that the question of lines and points introduced on conic diagrams would take significant thought to resolve.

So What Can We Do?

In the end, it may not be very useful to patch the *Conics* by exactly characterizing its deficiencies. The results Apollonius obtained still hold, and the proofs are a great insight into Greek geometry. However, the missing modes of inference are not too distinct from Euclidean concerns. While Euclidean geometry is still relevant (and taught in schools), and other works like those of Archimedes provide great insight into advanced Greek thinking, Apollonius's *Conics* have mostly served their time. Apollonius gave the conic sections their names, and documented their properties that were surely useful from pure mathematics to applications like astronomy, his proofs are no longer how we *think* about conic sections — one might sooner find a mathematician talking about *quadratic curves* instead of conic sections, indicating a bias towards algebra. Thus, we leave the *Conics* as an important work with its own shortcomings and achievements.

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