Math 152 Notes

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PNT: Probability n is prime: $\sim log(n)$. Define $\pi(x) = \#primes \leq x$; PNT: $\pi(x) \sim \int_2^x \frac{dx}{log(x)}$

Equivalent: $\displaystyle\sum_{p \leq x} log(p) \sim x$

Better: $\displaystyle\sum_{n=p^k \leq x} log(p) \sim x$

$\displaystyle\sum_{x \leq n} \Lambda(n) \sim x$ where $\Lambda(n) = (\delta_{n=p^k}) log(x)$ (von Mangoldt function)

$\displaystyle\sum \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}$

$\frac{1}{2\pi i} \displaystyle\sum_{\sigma - i\infty}^{\sigma + i\infty} x^s \frac{ds}{s} = (1, x > 1) | (0, x < 1) | (\frac{1}{2}, x = 1)$

$-\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\zeta(s)}{\zeta(s)} x^s \frac{ds}{s} = \psi(x)$

$\sigma > 1$, since ζ blows up at $s = 1$.

Remarkable, but typical complex analysis that this dose not depend on σ. We would like to move σ up to 1.

Key point to check: $\zeta(1 + it) \neq 0$ ($t \neq 0$ real)

If f is a continuous function on Ω (open subset of $\mathbb{C}$), f is called holomorphic or analytic if it has derivatives at all $a \in \Omega$

$f'(a) = lim_{k \to 0} \frac{f(a+k) - f(a)}{k}$ exists.

$k \to 0$ from any direction, so this is a strong assumption. Difference between real and complex analysis: If f is hol. it has derivatives of all orders.

If Ω is connected, a hol. function on Ω is determined by its values near any point: analytic continuation.

If f, g holomorphic on Ω, $f(x) = g(x)$ when $|x - a| < \epsilon \Rightarrow f = g$ on Ω.

Holomorphic functions on Ω form a ring, but not a field. To get a field...

f defined on Ω (except: allow a discrete set of exceptions called poles) is called meromorphic if $\forall a \in \Omega$ $f(x) = (x - a)^k f_1(x)$ (k could be pos. or neg.)

for x near a where $f_1(x)$ is holomorphic.

$\frac{1}{x^10}$ is meromorphic un $\mathbb{C}$ with a pole of order 10 at $x = 0$.

If $f(a) \neq 0$, then k is called the order of f at a

$k > 0$: zero of order k

$k < 0$: pole of order $|k|$

Meromorphic functions Ω form a field.
$\sqrt(x), log(x), e^{\frac{1}{x}}$ have singularities at 0 not poles.

$\zeta(s)$ is meromorphic on all of $\mathbb{C}$ with only one pole at $s = 0$.
If $\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx$ meromorphic on all $\mathbb{C}$, poles at $0, -1, -2, ...$

Theorem(Riemann): $Z(s) = \prod^{-\frac{s}{2}} \Gamma(\frac{s}{2})\zeta(s)$ is mero on all $\mathbb{C}$, vih poles at $s = 0, 1$ (no others) and $Z(s) = Z(1-s)$ (functional equation).

One Proof: Riemann
Define $\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$
From Fourier analysis, $\theta(t) = \frac{1}{\sqrt(t)}\theta(\frac{1}{t})$
$\frac{1}{2}(\theta(t) - 1) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}$
$\prod^{-\frac{s}{2}} \Gamma(\frac{s}{2})\zeta(s) = \int_0^\infty \frac{1}{2}(\theta(t)-1)t^{\frac{s}{2}}\frac{dt}{t}$, valid if $re(s) > 1$) (substitute series $\zeta(s) = \frac{1}{n^s}$, evaluate term by term).
$\prod^{-\frac{s}{2}} \Gamma(\frac{s}{2})\zeta(s) = \frac{1}{s} + \frac{1}{1-s} + \int_1^\infty \frac{1}{2}(\theta(t)-1)(t^{\frac{s}{2}} + t^{\frac{1-s}{2}})\frac{dt}{t}$ (convergent for all $s \neq 0, 1$)
$\prod^{-\frac{s}{2}} \Gamma(\frac{s}{2})\zeta(s) =$
$x - \sum_{\rho;\zeta(\rho)=0} \frac{x^\rho}{\rho} + \sum_{n=1}^{\infty} \frac{x^2 n}{2n} - log(2\pi)$ (von Mangoldt)
If the Riemann Hypothesis is true, $|\psi(x) - x| = O(x^{\frac{1}{2}\text{ish}})$
Proof: If $t \neq 0$ is real, $\zeta(1 + it) \neq 0$
$\sigma_a(n) = \sum_{d|n} d^a$
$\sum \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}$
Take $a = it, b = -it$
$\sum_{n=1}^{\infty} \frac{|\sigma_{it}(n)|^2}{n^s} = \frac{\zeta(s)^2\zeta(s-it)\zeta(s+it)}{\zeta(2s)}$
Assume $\zeta(1 + it) = 0$, $\zeta(1 - it) = \overline{\zeta(1 + it)} = 0$
Both $\zeta(s + it), \zeta(s - it)$ have zeros, cancel pole of order 2 of $\zeta(s)^2$ at $s = 1$
Denominator $\zeta(2s)$ is given by a convergent series in $Re(s) > \frac{1}{2}$ and is not zero. $\zeta(2s) = \prod_p (1 - \frac{1}{p^2 s})^{-1}$ (paren nonzero if $Re(s) > \frac{1}{2}$)
$H(s) = \sum \frac{|\sigma_{it}(n)|^2}{n^s}$
If $s = \frac{1}{2}$, $\zeta(2s)$ has a pole, so actually, $H(\frac{1}{2})$ (Ingham, 1930)

This is impossible...
Suppose $\sum \frac{a_n}{n^s}$ is any Dirichlet series.
If $\sigma_0 = inf\{\sigma | \sum \frac{a_n}{n^\sigma} < \infty\}$
Then σ_0 is the abscissa of convergence
$H(s) = \sum \frac{|\sigma_{it}|^2}{n^s}$ is holomorphic on $Re(s) > \frac{1}{2}$.
Series converges absolutely to the right of σ_0; diverges to left. σ_0 could be $-\infty$ to ∞. For $\zeta(s)$, $\sigma_0 = 1$
If a_n ae positive, $\sum \frac{a_n}{n^s} = H(s)$ is (for real s) monotone decreasing to the right of σ_0 if $\sigma_0 \neq \infty$.
σ_0 is a singularity (pole).
Since $H(s)$ is analytic to the right of $\frac{1}{2}$, $\sigma_0 \leq \frac{1}{2}$ and $lim_{s \to \frac{1}{2}} H(s) = 0$ is holomorphic.