Math 152 Notes

Lucas Garron

December 1, 2009

20091201

Lemma: Suppose $f(k) = \frac{\alpha_0^{k+1} - \alpha_1^{k+1}}{\alpha_0 - \alpha_1}, g(k) = \frac{\beta_0^{k+1} - \beta_1^{k+1}}{\beta_0 - \beta_1}$
Then $\sum_k = 0^\infty f(k)g(k)x^k = \frac{1 - \alpha_0\alpha_1\beta_0\beta_1 x^2}{(1-\alpha_0\beta_0 x)(1-\alpha_0\beta_1 x)(1-\alpha_1\beta_0 x)(1-\alpha_1\beta_1 x)}$

PNT essential thought: Probability of n being prime is about $1/log(n)$. Conjectured by Gauss, proved in 1896 by Hadamard and de Vallee-Poussin. Essential ideas for proof found by Riemann, Chebyshev. Riemann, 1859, gave a short powerful paper on the Riemann zeta function; contained ideas leading to a proof of the PNT.

If we believe the probability of n being prime is about $1/log(n)$, the number of primes $\leq x \approx \int_2^x \frac{dx}{log(x)}$

If $f(x), g(x)$ are functions, $f \sim g$ if $lim_{\rightarrow\infty} \frac{f(x)}{g(x)} = 1$

$\pi(x) = |\{p | p \text{ prime}, p \leq n\}| \sim \int_2^\infty \frac{dx}{log(x)} \sim \frac{x}{log(x)}$

$\Lambda(n) = log(p), n = p^k, \text{ else } 0$
$\psi(x) = \sum_{n \leq x} \Lambda(n)$. If PNT true, we expect $\psi(x) \sim x$. First prime powers p^k with $k > 1$ are sparse.
We can ignore. $\psi(x)$ is about $\sum_{n \leq x \text{ prime}} log(n) \sim x$ if probability of being prime is $1/log(n)$.

Fairly elementary: $\psi \sim x$ is equivalent to PNT.

$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda}{n^s}$

Proof: Remember ζ as product, $\frac{\zeta'(s)}{\zeta(s)} = \frac{d}{ds}\log(\zeta(s)) = \sum_p \frac{d}{ds} log(1-p^s)^{-1}$, $\frac{d}{ds}p^{-s} = log(p)p^{-s}$

$\frac{\zeta'(s)}{\zeta(s)} = -\frac{\log(p)p^{-s}}{1-p^{-s}}$
$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{log(p^{-s})}{1-p^{-s})} = \sum_p \left(p^{-s} + p^{-2s} + p^{-3s} + ...\right)$
$= \sum_n \frac{log(n)}{n} \delta_n \text{ prime}$

$\sigma > 0$: $\int_{\sigma-i\infty}^{\sigma+i\infty} x^s \frac{ds}{s} = \delta_{x>1}$
$\int_{-\infty}^{\infty} x^{\sigma+it} \frac{idt}{\sigma+it}$, $(s = \sigma + it,\ ds = idt)$
Remarkably, this does not depend on σ (Cauchy's theorem, typical complex analysis.)
As a function of x, discontinuous (typical Fourier analysis)

$\psi(x) = \int_{\sigma-i\infty}^{\sigma+i\infty} (\frac{\zeta'(s)}{\zeta(s)}) x^s \frac{ds}{s}$ (valid, proved for $\sigma > 1$)

Follows because $RHS = \sum_n \Lambda(n) \int_{\sigma-it}^{\sigma+it} (\frac{x}{n})^s \frac{ds}{s} = \sum_{x \leq n} \Lambda(n)$ (big thingy is $\delta_{x\leq n}$)

Key fact about ζ needed to deduce PNT. Hadamard, DLVP: If $t \neq 0$ is real, $\zeta(1+it)$ is finite, non-zero.
$\frac{\zeta'(-)}{\zeta(s)}$ blows up at $s = 0$ but nowhere else on line $1 + it$ $(treal)$, so we can move path of integration.
End up with $\int_{1-i\infty}^{1+i\infty}$, detour near singularity at $_{s=1}$ ... $= \psi(x)$ gives $\psi(x) \sim x$