Math 152 Notes

Lucas Garron

November 19, 2009

20091119

Thm: If $GCD(m,a)=1$, $\exists\infty$ many primes of the form $mk+a$

Thm: $\sum \frac{1}{p} = \infty$ $log(\zeta(s)) = -\sum_p log(1-\frac{1}{p^2}) = \sum_p \sum_{n=1}^{\infty} \frac{p^{-ns}}{n}$

Study behavior at $s=1$. Assume $s\in\mathbb{R}, s>\frac{1}{2}$, $s'=2s$

Claim: $\sum_p \sum_{n=2}^{\infty} \frac{p^{-ns}}{n}$ converges.

This is dominated by $\sum_p \sum_{n=2}^{\infty} p^{-ns}$

The exponents $ns>1$

$p^{-2s}=p^{-s'}, p^{-3s}=p^{\frac{3}{2}s'}$

This is bounded by $\sum_{m=2}^{\infty} m^{-s'} + \sum_m m^{-\frac{3}{2}s'} + ...$

$(\zeta(s')-1)+(\zeta(\frac{3}{2}s')-1)+(\zeta(2s')-1)$

If $f(s), g(s)$ two functions defined to the right of zero. $f\sim g$ means $f-g$ has a finite limit as $s\to 1$ (from the right)

$log(\zeta(s)) \sim \sum_p \frac{1}{p^s}$

Can show $\zeta(s)\sim\frac{1}{s-1}$

Sufficient for now $lim_{s\to 1^+}\zeta(s)=\infty$

$\zeta(s)=\sum\frac{1}{n^s}$ is compared to $\int_1^\infty \frac{1}{x^s}dx$ in integral test.

... follows $log(\zeta(s))\to\infty$ as $s\to 1+$, so $\sum\frac{1}{p^s}\to\infty$, so $\sum\frac{1}{p}$ diverges (Euler).

Dirichlet character mod4

$\chi(n)$ depends on n mod 4

$\chi(mn)=\chi(m)\chi(n)$, $\chi(n)=0$ unless $GCD(n,4)=1$

$$L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-ns}=\prod_p(1-\frac{\chi(p)}{p^s})^{-1}$$

As $s\to 1$, $L(s,\chi)$ does not blow up, nor tends to 0.

$L(1,\chi) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}$ (Gregory series)

(Thm): $lim_{s\to 1+} L(s,\chi)$ is a finite, nonzero number.
Lemma: Suppose a_n is a series of positive numbers s.t. $a_1 > a_2 > ... > 0$, $a_n \to 0$. Then $a_1 - a_2 + a_3 - a_4 + ...$ is convergent.

Thm (Dirichlet): If χ is a Dirichlet character mod$m$ (not "trivial"), then $L(1,\chi)$ is finite, nonzero.
How this can be proved:
$\zeta(s)L(s,\chi) = Z(s)$
$Z(s) = \frac{1}{4}\sum_{a+bi\neq 0} \frac{1}{(a^2+b^2)^s} = \sum_{\text{ideals I of } \mathbb{Z}[i]} \frac{1}{NI^s}$
$\zeta(s) = \frac{1}{s-1}, Z(s) = \frac{constant}{s-1}$
If $L(s,\chi) \to 0$ as $s \to 1$, $Z(s)$ would not blow up.

Proof $\exists\infty$ many primes $\equiv 1$ mod 4 (or $\equiv 3$)
$log(\zeta(s)) = \sum \frac{p^{-ns}}{n} \sim \sum p^{-s}$
$log(s,\chi) = -\sum_p log(1 - \frac{\chi(p)}{p^s})^{-1} = \sum_p \sum_{n=1}^{\infty} \frac{\chi(p)^n}{np^{ns}} \sim \sum_p \frac{\chi(p)}{p^s}$
$log(\zeta(p^s))$ blows up.
$log(L(s,\chi))$ does not at $s=1$; in fact $\to log(\pi)$
$log(s)$ blows up at $s=0$, which is why we need $L(1,\chi)\neq 0$
$\sum_{p\equiv 1} \frac{1}{p^s}$
$\frac{1}{2}\left(\sum \frac{1}{p^s} + \sum \frac{\chi(p)}{p^s}\right)$
So $lim_{s\to 1+} \sum_{p\equiv 1} \frac{1}{p^-} = \infty$

Thm (Dirichlet): If $GCD(a,m)=1$, $\exists\infty p \equiv a$ mod m
There are $\phi(m)$ Dirichlet characters χ mod m
(Assume for simplicity m prime:)
$\zeta(s) \prod_{\chi \bmod m, \chi \text{ nontrivial}} L(s,\chi) = \sum_{\text{Ideals in } \mathbb{Z}[e^{\frac{2\pi i}{m}}]} \frac{1}{NI^s}$ (Dedekind zeta function of $\mathbb{Z}[e^{\frac{2\pi i}{m}}]$ cyclotomic ring) - related to enumeration of ideals in $\frac{\mathbb{Z}[e^[2\pi i}{m]]}$

$zeta(s), Z(s)$ both blow up (poles of order 1)
$L(s,\chi)$ does not blow up.
Follows $L(1,\chi)\neq 0$
$\sum_{p\equiv a \bmod m} \frac{1}{p^s} \sim \frac{1}{\phi(m)}\left(log(\zeta(s)) + \sum_{\chi \text{nontrivial} \bmod m} \chi(a)^{-1} log(L(s,\chi))\right)$
Fix a, $a, m = 1$ $\frac{1}{\phi(m)} \sum_{\chi \bmod m} \chi(a)^{-1} \sum_{b \bmod m} \chi(b) = \delta_{a\equiv b \bmod m}$

$$\frac{1}{\phi(m)}\sum_{\chi}\chi(a)^{-1}\sum_{b \bmod m,(b,m)=1}\chi(b)\sum_{p\equiv b \bmod m}\frac{1}{p^s}=\sum_{p\equiv a(m)}\frac{1}{p^s} \text{ (orthogonality).}$$