Math 152 Notes

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Missed: First 15min
Things posted online.

Theorem: If f is multiplicative, so is $g(n) = \sum_{d|n} f(d)$
$f(n) = 1, g(n) = \#divisors(n)$
$f(n) = n, g(n) = \sum_{d|n} = \sigma(n)$
d, σ multiplicative.

Generalization: Suppose f_1, f_2 given.
$g(n) = \sum_{d|n} f_1(d) f_2(\frac{n}{d}) = \sum_{d|n} f_1(\frac{n}{d}) f_2(d)$ ("Dirichlet Convolution")
Theorem: If f_1, f_2 are multiplicative, so is $g(n) = \sum_{d_1 d_2 = n} f_1(d_1) f_2(d_2$
Proof: m, n coprime, $d|mn$ means $d = uv$, $u|m$, $v|n$ (u, v coprime)
$g(mn) = \sum_{d|mn} f_1(d) f_2(\frac{mn}{d}) = \sum_{u|m, v|n} f_1(uv) f_2(\frac{mn}{uv})$
$= \sum_{u|m, v|n} f_1(u) f_1(v) f_2(\frac{m}{u}) f_2(\frac{n}{v})$
$= \sum_{u|m} f_1(u) f_2(\frac{m}{u}) \sum_{v|n} f_1(v) f_2(\frac{n}{v}) = g(m) g(n)$
Mobius μ
Proposition: $\sum_{d|n} \mu(d) = \delta_{n,1}$
Proof: μ is mult. So $g(n) = \sum_{d|n} mu(d)$ is mult., and its's sufficient to check $g(p^k) = 0$ if $k > 0$, p prime
$\sum_{d|p^2} \mu(p^k) = \mu(1) + \mu(p) + ... + \mu(p^k) = 1 + (-1) + 0 + ... + 0 = 0$
Moebius inversion formula: If $g(n) = \sum_{d|n} f(d)$, then $f(n) = \sum_{d|n} \mu(d) g(\frac{n}{d}) = \sum_{d|n} g(d) \mu(\frac{n}{d})$

$f = \mu * g$

Proof
$\sum_{d|n} g(d)\mu(\frac{n}{d}) = \sum_{d|n}\sum_{t|d} f(t)\mu(\frac{n}{d})$
Any t is a divisor of n: $t|d|n$.
$\sum_{t|n}\sum_{d|n,t|d} \mu(\frac{n}{d})$
$\frac{n}{t} = \frac{n}{d} \cdot \frac{d}{t} = u \cdot v$
Claim: u, v can be any divisors of $\frac{n}{t}$.
$u|\frac{n}{t}, v = \frac{n/t}{u}, t|\frac{n}{u}$, both say $\frac{n}{ut} \in \mathbb{Z}$
$= \sum_{t|n} f(t) \sum_{uv=\frac{n}{t}} \mu(u) = \sum_{t|n} f(t)\delta_{\frac{n}{t}=1}$
Applications: $\sum_{d|n} \phi(d) = n \Rightarrow \sum_{d|n} \mu(d)\frac{n}{d} = \sum_{d|n} \mu(\frac{n}{d}) \cdot d = \phi(d)$
$\phi(n) = n\sum_{d|n} \frac{\mu(d)}{d}$
$d(n) = \#divisors(n) = \sum_{d|n} 1$
$1 = \sum_{d|n} \mu(d)d(\frac{n}{d})$

8.2 Dirichlet Series
$\sum \frac{a_n}{n^s}$
Ted to converge in right-half planes.
If $\sum \frac{a_n}{n^s}$ converges for $s = \sigma + it$, it converges anywhere to the right.
Abscissa of convergen.
$a_n = 1$: $\frac{\sum 1}{n^s} = \zeta(s)$, abs. conv. if $\sigma > 1$
$f(s) = \sum \frac{a(n)}{n^s}$, $b(n) = \sum_{d|n} a(d)$
Then $\sum \frac{b(n)}{n^s} = \zeta(s)\sum \frac{a(n)}{n^s}$ because $\zeta(s)\sum \frac{a(n)}{n^s} = \sum_m \frac{1}{m^2}\sum_n \frac{a(n)}{n^s}$
$\sum_{m,n} \frac{a(n)}{mn^s} = \sum_M \frac{1}{M^s}\sum_{M=nm} a(n)$
$= \sum_M \sum_{n|M} a(n) = \sum_M \frac{b(M)}{M^s}$
If a_1, a_2 giwv