Math 152 Notes

Lucas Garron

November 10, 2009

20091110

$R(n) = 4\sum_{d|n} \chi(d)$
$\chi(n) = 0 \to 0, 1 \to 1, 0 \to 2, -1 \to 3$
$R(n) = \#$ of reps of n by $x2 + 5y^2$ or $2x^2 + 2xy + 3y^2$
Last time classified Gaussian primes.
Recall a prime in $R = \mathbb{Z}[i]$ is an irreducible element, except that associates determine the same prime.
Alternatively, let $\alpha \in R$, $\alpha = R\alpha =$ all multiples of α. Only depends on class of associates of α.
$R = \mathbb{Z}[i] \subset \mathbb{Q}(i) = \mathbb{Q}[i]$ A prime is an ideal represented by an irreducible. (Actually, I prime means that R/I is an integral domain - for $\mathbb{Z}$ or $\mathbb{R}$, this would be a field.)
Failing factorization remedied by looking at IDEALS.
$2 = I^2, 3 = P_1 P_2$
$IP_1 = (I + \sqrt{-5}), IP_2 = I - \sqrt{5}$

Gaussian primes: $1 + i$, norm 2
$p \equiv 1 \bmod 4$ $p = a^2 + b^2$ (Fermat) $\Rightarrow a + bi, a - bi$ primes.
$p \equiv 3 \bmod 4 \Rightarrow p$ remains prime in R. (one prime of norm p)

$\zeta(s) = \eta(s) = \sum \frac{1}{n^2} = \prod_p (1 - \frac{1}{p^2})^{-1}$ (Euler Product)
Everything convergent if $s > 1$

Generalization Time!
$L(s, \chi) = \sum_n \frac{\chi(n)}{n^2} = \prod_p (1 - \frac{\chi(p)}{p^s})^{-1}$
(Useful for proving infinitely many primes within arithmetic progressions.)
Let $Z(s) = \sum_{\text{Ideals of } R} \frac{1}{N(I)^s} = \prod_{\text{prime ideals } \pi} \left(1 - \frac{1}{N(\pi)^s}\right)^{-1}$
This is the Dedekind zeta function.
Proof same: Use unique factorization in R.
$Z(s) = \frac{1}{4} \sum_{a,b \in \mathbb{Z}, \text{ not both } 0} \frac{1}{(a^2 + b^2)^s}$
(The ideal I can be represented as $a + bi$ in four ways.)

$Z(s) = \frac{1}{4}\sum_n \frac{R(n)}{n^s}$

$$\prod_{\text{prime Ideals P of R}} (1 - \frac{1}{NP^s})^{-1}$$

$Z(s) = (1 - \frac{1}{2^s})^{-1} \left(\prod_{p \equiv 3 \bmod 4} (1 - \frac{1}{p^s})^{-1} \right) \left(\prod_{p \equiv 1 \bmod 4} (1 - \frac{1}{p^{2s}})^{-1} \right)$

$Z(s) = (1 - \frac{1}{2^s})^{-1} \left(\prod_{p \equiv 1 \bmod 4} (1 - \frac{1}{p^s})^{-1} (\chi(p) - \frac{1}{p^s})^{-1} \right) \left(\prod_{p \equiv 3 \bmod 4} (1 - \frac{1}{p^s})^{-1} (\chi(p) - \frac{1}{p^s})^{-1} \right)$

$= \prod_p (1 - \frac{1}{p^s})^{-1} \prod_p (\chi(p) - \frac{1}{p^s})^{-1} = \zeta(s) L(s, \chi)$

$\frac{1}{4}\sum_n \frac{R(n)}{n^s} = Z(s) = \zeta(s) L(s, \chi)$

$= \sum_{d,k} \frac{\chi(d)}{(kd)^s}$

$= \sum_n \frac{1}{n^2} \sum_{d|n} \chi(d)$ Compare coefficients of

Moebius function, Moebius inversion

Often we sum over divisors of n.

$f : \mathbb{Z}^+ \to \mathbb{C}$

$g(n) = \sum_{d|n} f(d)$, e.g. $\sum_{d|n} \phi(d) = n$

Theorem: If $g(n) = \sum_{d|n} f(d)$, then $f(n) = \sum_{d|n} \mu(d)(\frac{n}{d}) = \sum_{d|n} \mu(\frac{n}{d}) \cdot g(d)$

where $\mu(d) = 0$ unless d is squarefree, $(-1)^k$ if d is a product of k distinct primes.

$\mu(1) = -1, 2 \to -1, 3 \to -1, 4 \to 0, 5 \to -1, 6 \to 1$

$\phi(6) = 6 - 3 - 2 + 1 = 2$

Proof: Consider $\sum \frac{f(n)}{n^s}$ (hope it converges)

$\zeta(s) \sum \frac{f(n)}{n^s} = \left(\sum_k \frac{1}{k^s} \right) \left(\sum_d \frac{f(d)}{d^s} \right) = \sum_{d,k} \frac{f(d)}{(kd)^s} = \sum_n \frac{1}{n^s} \sum \frac{1}{n^2} \sum_{d|n} f(d) = \sum \frac{1}{n^s} g(n)$

$\zeta(s) = \prod_p (1 - \frac{1}{p^s})$, expand:

$\sum \frac{\mu(n)}{n^s}$

$\sum \frac{f(n)}{n^s} = \frac{1}{\zeta(s)} \sum \frac{g(n)}{n^s} = \left(\sum_{d=1}^{\infty} \frac{\mu(d)}{d^s} \right) \left(\sum_{k=1}^{\infty} \frac{g(k)}{k^s} \right)$

$= \sum_n \frac{1}{n^s} \sum_{k,d} \mu(d) g(k) = \sum_N \left(\sum_{d|n} \mu(d) g(\frac{n}{d}) \right) n^{-s}$

μ, like ϕ, is multiplicative.

Often, but not always, Moebius inversion is applied where f is multiplicative.
Lemma: f, g multiplicative $\Rightarrow$ so is $h(n) = \sum_{d|n} f(d)g(\frac{n}{d})$. ("Convolution")

(If in MIF (Moebius inversion formula), g is multiplicative $\Leftrightarrow$ f is multiplicative.)