Math 102 (♣♦♥♠ Mathematics & Magic) Course Notes April 05, 2012

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These notes are available at http://stanford.garron.us/class/math102/

1 Meta

- Course notes from the first day are on Coursework, under Syllabus.
- Today is the last day for auditions.

2 Group Theory

2.1 Definition of a Group

A group is

- a set G,
- and a notion of multiplication, $G \times G \to G$, so $(g_1, g_2) \to g_1 * g_2$

The following have to hold:

1. The product has to be associative.

$$(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$$

2. THere must be an identity, which we'll write as $id = 1$, such tha

$$1 * g = g * 1 = g$$

3. Every element has an inverse in G. The inverse of g is written g^{-1}, so that

$$g^{-1} * g = g * g^{-1} = 1$$

Closure

2.2 Example: Cyclic Groups

$$C_n = \{0, 1, 2, ..., n-1\}$$

Multiplication is defined by

$$i * j = i + j \bmod n$$

(e.g. $n = 5$, $4 + 4 \equiv 3 \bmod 5$)

2.2.1 Example of a cyclic group: Cutting a Deck

n cards form C_n under the operations of cutting cards i from top to bottom.

$id = i$, $i^{-1} = -i$

3 Permutation Groups

$$S_n = \{\text{All arrangements of a deck of } n \text{ cards labeled } 1, 2, 3, ..., n\}$$

Notation: $\pi(i)$ is the label of the card at position i.

3.1 Example of a permutation

$$\pi = 3\ 2\ 1\ 4\ 5$$

Card 3 on top, card 2 below, etc.

3.2 Long Notation

We can also write this as

$$\begin{pmatrix} 1 & 2 & 3 & ... & n \\ \pi(1) & \pi(2) & \pi(3) & ... & \pi(n) \end{pmatrix}$$

For example, switching the top two cards is the following permutation:

$$x = \begin{pmatrix} 1 & 2 & 3 & ... & n \\ 2 & 1 & 3 & ... & n \end{pmatrix}$$

Cutting the top to bottom:

$$y = \begin{pmatrix} 1 & 2 & 3 & ... & n \\ 2 & 3 & 4 & ... & 1 \end{pmatrix}$$

The product $\pi * \sigma$ is defined as doing σ, and then doing π. Think of the permutation like a function: $\pi * \sigma = \pi(\sigma)$ or $a * b * c * d = a(b(c(d)))$

Using x and y as defined above, we have

$$x * y = \begin{pmatrix} 1 & 2 & 3 & ... & n \\ 3 & 2 & 4 & ... & 1 \end{pmatrix}$$

But x and y don't commute, so the order of multiplication matters:

$$y * x = \begin{pmatrix} 1 & 2 & 3 & ... & n \\ 1 & 3 & 4 & ... & 2 \end{pmatrix}$$

3.3 Inverse of a Permutation

If

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \pi(1) & \pi(2) & \pi(3) & \dots & \pi(n) \end{pmatrix},$$

then the inverse π^{-1} can be computed like this:

1. Find 1 in row 2; whatever is above is $\pi^{-1}(1)$.
2. Find 2 in row 2; whatever is above is $\pi^{-1}(2)$.
3. ...

Alternatively, you could think of it this way:

1. Switch the two rows.
2. Sort the top row.

Thus,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$$

3.4 Perfect Riffle Shuffles

If you split a deck of 8 cards perfectly in half and riffle shuffle them (alternate between dealing from halves to form a pile), the following happens:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 2 & 6 & 3 & 7 & 4 & 8 \end{pmatrix}$$

3.5 Element Order

The *order* of any group element g is the smallest k such that

$$\underbrace{g * g * g * \dots * g}_{k} = 1$$

3.5.1 Example: Cut 1 card

If $G = S_n$ and $\sigma =$ (move a card from top to bottom), then the order of σ is n.

3.5.2 Order of a Riffle Shuffle

Problem: Find the order of a perfect shuffle of $2n$ cards, e.g. how many perfect riffle shuffles do you need to do to a deck of cards until you get back to the original order?

Note: Although this is easy to compute for any n, there is no general known solution for finding the order.

3.6 Cycle Notation

Given π:

1. Write down a cycle
   (a) Write down the smallest number not used (e.g. 1 for the cirst cycle); call it x
   (b) Write down $\pi(x)$.
   (c) Write down $\pi(\pi(x))$.
   (d) Stop when (right before) you get back to x.
2. If there are any numbers left, write down another cycle.

3.6.1 Cycle Notation Example

For example, take

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 7 & 2 & 5 & 1 & 8 & 4 & 10 & 9 & 6 \end{pmatrix}$$

In cycle notation, this is

$$(1\ 3\ 2\ 7\ 4\ 5)(6\ 8\ 10)$$

(Note that we left out 9).

From cycle notation, we see that a cycle recycles to its original position with an order that is its length.

Say we have (123)(5746). The first part takes 3 repetitions to recycle, but the second part takes 4 repetitions. They both recycle together after 12 repetitions.

Proposition 1 *In general, if σ has cycles of length $l_1, l_2, ..., l_n$, the order of σ is $lcm(l_, l_2, ..., l_n)$.*

3.6.2 What shuffle takes the longest?

What permutation (of S_n) takes the longest number of repetitions to recycle?

We want distinct (relatively) prime parts that add up to n but have the greatest product.

Example: For 52 cards, this is 180180.

3.7 Antimagic Squares

3.7.1 A Magic Square

Here's a $3x3$ magic square. This is a square grid with number such that every row, column, and diagonal adds up to the same number (15 in this case).

4	3	8
9	5	1
2	7	6

A mnemonic for reconstructing this: Put $4, 5, 6$ on the diagonal, then fill in the rest.

3.7.2 An Antimagic Square

Instead of adding up to the same sum, we want all the directions to add up to *different* sums:

1	2	3
8	9	4
7	6	5

(It's a spiral!)

3.7.3 An Antimagic Square Trick

Lay out the cards in an anti magic square. Give someone in the audience a prediction, and give the directions to someone else.

- Put your finger on any card.
- Look at the number of that card, and hop with your finger that often (each hop is to an adjacent card, horizontally or vertically).
- Remove the ace.
- Hop 4 times, remove the 5.
- Hop 3 times, remove the 2.
- Hop 3 times, remove the 3.
- Hop 5 times, remove the 4.
- Hop 4 times, remove the 6.
- Hop 1 times, remove the 7.
- Hop 3 times, remove the 8.
- You card was the 9 (e.g. 9♦ if that was the 9 on the table).

3.7.4 History

Persi came up with this trick when he was young, and showed it to Martin Gardner; it's been used to great effect (e.g. commercialized as "Voice from Another Planet") over the years.

4 Parity

4.1 A Trick

Lay out 5 cards: 1 2 3 4 5

- Put your finger on any card, remove the ace.
- Move 2 more steps, remove the 5.

- Move 3 more steps, remove the 2 and the 4.
- Your finger is on the 3.

4.1.1 Analysis

After the first move, you're on one of the even cards (ace is safe to remove). Then, if you move any even number of steps, you have to be back on an even-numbered card (5 is safe to remove). Then, if you move any odd number of steps, you have to end up on the 3.

There are two ideas at play:

- If you're on an even position, then
    1. an even number of hops will keep you at an even position.
    2. an odd number of hops will switch you to an odd position

  (and complementarily for starting at an odd position).
- If the original layout is 1 2 3 4 5, we can force the parity to an even position "naturally".

4.2 Martin Gardner's "13th Turn"

1. Place a die on the table, look at a corner, and add up the dots touching it.
2. Look away, and ask a volunteer to to tip the die of over an edge 12 time, and then choose to do it a 13th time or not (their choice).
3. Ask them to tip it 4 more times.
4. Look at the die and announce whether they did the 13th turn earlier.

The trick: The parity of the dots around a corner changes every roll.

4.3 Informally Definition of Parity

"Subgroups of order 2".